Mathematics 222B Lecture 17 Notes

Daniel Raban

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1 Local Well-Posedness of the Initial Value Problem for Variable-Coefficient Wave Equations

1.1 Recap: setting and statement of the estimate

We have been looking at linear hyperbolic PDEs $P\phi = f$, where

$$P\phi = \partial_\mu(g^{\mu,\nu}\partial_\nu \phi) + b^\mu \partial_\mu \phi + c\phi.$$

We want to solve the initial value problem

$$\begin{cases} P\phi = f \\ (\phi, \partial_t \phi)|_{t=0} = (g, h). \end{cases}$$

To discuss existence and uniqueness, we made further assumptions on the coefficients:

- $g^{\mu,\nu}$ is a symmetric $(1+d)\times(1+d)$ matrix with signature $(-,+,+,\dots,+)$.
- $g^{0,j}(t,x) = 0$ and $g^{0,0}(t,x) = -1$.
- For $\xi \in \mathbb{R}^d$, $g^{j,k}\xi_j\xi_k \geq \lambda|\xi|^2$ (bottom right $d \times d$ minor is positive definite).
- $g^{\mu,\nu}, b, c$ are uniformly bounded, with uniformly bounded derivatives.

Example 1.1. Set $b = c = 0$, and let $g = \mathrm{diag}(-1,1,1,\dots,1)$. Then $P = \Box$.

We take the convention that $x^0 = t$. We also use Greek indices $\mu, \nu \in \{0,1,\dots,d\}$ and indices $j,k \in \{1,\dots,d\}$. Last time, we were proving the following theorem.

Theorem 1.1 (Local well-posedness of the initial value problem)**.** *Let $s \in \mathbb{Z}_+$. Given $(g,h) \in H^{s+1} \times H^s(\mathbb{R}^d)$ and $f \in L^1_t([0,t];H^s(\mathbb{R}^d))$, there exists a unique solution ϕ to the initial value problem with $\phi \in C_t([0,T],H^{s+1})$ and $\partial\phi \in C_t((0,T);H^s)$. Moreover, the unique solution ϕ satisfies the estimate*

$$\|\phi\|_{C_t([0,T];H^{s+1})} + \|\partial_t \phi\|_{C_t([0,T];H^s)} \lesssim_{g^{\mu,\nu},b^\mu,c,T,s} \|(g,h)\|_{H^{s+1}\times H^s} + \|f\|_{L^1_t([0,T];H^s)}.$$

Remark 1.1. Local well-posedness entails continuous dependence of ϕ on (f,g,h). Because of linearity, this a priori estimate implies continuous dependence (and in fact Lipschitz dependence).

1.2 Proof of the a priori estimate

Let's finish the proof. Recall that the idea of the proof is to use the a priori estimate, along with a functional analytic lemma.

Proposition 1.1. *Let $s \in \mathbb{Z}$. Let $\phi \in C_t([0,T]; H^{s+1})$ and $\partial_t \phi \in C_t([0,T]; H^s)$. Then*

$$\|\phi\|_{C_t([0,T];H^{s+1})} + \|\partial_t \phi\|_{C_t((0,t):H^s)} \lesssim \|(\phi, \partial_t \phi)|_{t=0}\|_{H^{s+1} \times H^s} + \|P\phi\|_{L^1_t([0,T];H^s)}.$$

Proof. ($s \geq 0$): We want to use the energy method. The natural strategy would be to commute $P\phi$ with D^α for $|\alpha| \leq s$ and apply the energy estimate (multiply by $\partial_t \phi$ and integrate by parts). Instead, we vary the multiplier:

$$\langle P\phi, (1-\Delta)^s \partial_t \phi \rangle := \int P\phi (1-\Delta)^s \partial_t \phi \, dx$$

- On one hand, we know by duality that

  $$\int_0^T \langle P\phi, (1-\Delta)^s \partial_t \phi \rangle \, dt \lesssim \|P\phi\|_{L^1_t([0,T];H^s)} \|\partial_t \phi\|_{C_t([0,T];H^s)}.$$

  This is basically integrating by parts s times and using Cauchy-Schwarz. We can also think of this as the general bound

  $$|\langle f, g \rangle| \lesssim \|f\|_{H^s} \|g\|_{H^{-s}}$$

  In general, if Q is an order r differential operator with that have uniformly bounded derivatives to all order, then (with some Fourier analysis), we can say that

  $$\|Qg\|_{H^s} \lesssim \|g\|_{H^{r+s}} \qquad (s \in \mathbb{R}).$$

  For negative s, we get the inequality by duality:

  $$\begin{aligned} \|Qf\|_{H^s} &= \sup_{\|g\|_{H^s}=1} ||\langle Qf, g \rangle| \\ &= \sup_{\|g\|_{H^s}=1} ||\langle f, Q^* g \rangle| \\ &\lesssim \|f\|_{H^{s+r}} \|Q^* g\|_{H^{s-r}}. \end{aligned}$$

  We also have the fact that

  $$\|(1-\Delta^s) g\|_{L^2} \simeq \|g\|_{H^{2s}}, \langle (1-\Delta)^s g, g \rangle \simeq \|g\|^2_{H^s},$$

  which we get by using the Fourier transform:

  $$\langle (1-\Delta)^s g, g \rangle = \langle (1+|\xi|^2)^s, \widehat{g}, \widehat{g} \rangle = \|(1+\xi|^2)^{s/2} \widehat{g}\|^2_{L^2}$$

- On the other hand, we have

$$P\phi = \underbrace{\partial_\mu(g^{\mu,\nu}\partial_\nu\phi)}_{-\partial_t^2\phi+\partial_j(g^{j,k}\partial_k\phi)} + b^\mu\partial_\mu\phi + c\phi.$$

Now we can observe that

$$\langle -\partial_t^2\phi, (1-\Delta)^s\partial_t\phi\rangle = -\partial_t\langle\partial_t\phi, (1-\Delta)^s\partial_t\phi\rangle + \langle\partial_t\phi, (1-\Delta)^s\partial_t^2\phi\rangle$$

Since $\langle\partial_t\phi, (1-\Delta)^s\partial_t^2\phi\rangle = \langle(1-\Delta)^s\partial_t\phi, \partial_t^2\phi\rangle$, we get

$$= -\frac{1}{2}\partial_t\langle\partial_t\phi, (1-\Delta)^s\partial_t\phi\rangle$$

For the other term, we have

$$\begin{aligned}\langle\partial_j(g^{j,k}\partial_k\phi), (1-\Delta)^s\partial_t\phi\rangle &= -\langle g^{j,k}\partial_k\phi, (1-\Delta)^s\partial_t\partial_j\phi\rangle \\ &= -\partial_t\langle g^{j,k}\partial_k\phi, (1-\Delta)^s\partial_j, \phi\rangle \\ &\qquad + \langle\partial_t g^{j,k}\partial_k\phi, (1-\Delta)^s\partial_j, \phi\rangle \\ &\qquad + \langle g^{j,k}\partial_k\partial_t\phi, (1-\Delta)^s\partial_j\phi\rangle.\end{aligned}$$

Write the last term as

$$-\langle_t\phi, \partial_k(g^{j,k}(1-\Delta)^s\partial_j\phi)\rangle = -\langle\partial_t\phi\partial_k([g^{j,k}, (1-\Delta)^s]\partial_j\phi)\rangle \underbrace{-\langle\partial_t\phi, \partial_k(1-\Delta)^s(g^{j,k}\partial_j\phi)\rangle}_{=-\langle(1-\Delta)^s\partial_t\phi, \partial_k(g^{j,k}\partial_j\phi\rangle}.$$

Overall, this equals

$$-\frac{1}{2}\partial_t\langle g^{j,k}\partial_k\phi, (1-\Delta)^s\partial_j\phi\rangle + \frac{1}{2}\langle\partial_t g^{j,k}\partial_k\phi, (1-\Delta)^s\partial_j\phi\rangle - \frac{1}{2}\langle\partial_t\phi, \partial_k([g^{j,k}, (1-\Delta)^s]\partial_j\phi)\rangle.$$

The point is of this messy calculation is as follows: for the terms with the highest number of derivatives, we want to put things in to this total derivative form. The other terms will have at least 1 derivative that is not falling on ϕ. This is the purpose of using the commutator. What we get is that

$$\begin{aligned}&\langle P\phi, (1-\Delta)^s\partial_t\phi\rangle \\ &\quad = \underbrace{-\frac{1}{2}\partial_t(\langle\partial_t\phi, (1-\Delta)^s\partial_t\phi\rangle + \langle g^{j,k}\partial_k\phi, (1-\Delta)^s\partial_j\phi\rangle)}_{E_s[\phi](t)} \\ &\quad + \underbrace{O(\langle q_1\partial\phi, \partial^{2s}\partial\phi\rangle) + O(\langle q_2\partial\phi, \partial^{2s-1}\partial\phi\rangle) + \cdots + O(\langle q_{2s+1}\partial\phi, \partial\phi\rangle)}_{R_s},\end{aligned}$$

where $q_1 = \partial g, b$, $q_2 = \partial^2 g\partial bc$, etc.

So our energy argument says

$$\int_0^t \langle P\phi, (1-\Delta)^s \partial_t \phi \rangle \, dt' \geq E_s[\phi](0) - E_s[\phi](t) - C \int_0^t \|\phi\|_{H^{s+1}}^2 + \|\partial_t \phi\|_{H^s}^2 \, dt',$$

where we are just using the estimate for the remainder:

$$|R_s(t') \lesssim (\|\phi\|_{H^{s+1}} + \|\partial_t \phi\|_{H^s})^2.$$

Now we have

$$E_s[\phi](t) \leq E_s[\phi](0) + \|P\phi\|_{L_t^1([0,T];H^s} \|\partial_t \phi\|_{C_t((0,T);H^s)} + \int_0^t \|\phi\|_{H^{s+1}}^2 + \|\partial_t \phi\|_{H^s}^2 \, dt'.$$

Note that $E_s[\phi](t) \simeq \|\phi\|_{H^{s+1}}^2 + \|\partial_t \phi\|_{H^s}^2$, so our proprties of H^s and the elliptic estimate for $\partial_j g^{j,k} \partial_k$ gives:

$$E_s[\phi(t) \leq E_s[\phi](0) + \|P\phi\|_{L_t^1([0,T];H^s)} \|\partial_t \phi\|_{C_t((0,T);H^s)} + \int_0^t E_s[\phi(t') \, dt'$$

So Grönwall's inequality tells us that

$$E_s[\Phi](t) \lesssim E_s[\phi](0) + \|P\phi\|_{L_t^1([0,T];H^s)} \sup_{t \in [0,T]} E_s[\phi](t).$$

Now we can take the sup over $t \in [0,T]$ on the left hand side and use the AM-GM inequality with an epsilon to absorb the $\sup_{t \in [0,T]} E_s[\phi](t)$ on the right into the left hand side.

($s < 0$): Let $\Phi = (1-\Delta)^{-|s|} \phi$. We have the equivalence

$$\|\Phi\|_{H^{|s|+1}} \simeq \|\phi\|_{H^{-|s|+1}} = \|\phi\|_{H^{s+1}}.$$

Similarly,

$$\|\partial_t \Phi\|_{H^{|s|}} \simeq |\partial_t\|_{H^s}.$$

Now, we do the same argument with s replaced by $|s|$ and ϕ replaced by Φ. The only thing that is different is part 1 above. So we need to estimate

$$\begin{aligned} |\langle P\Phi, (1-\Delta)^{|s|} \partial_t \Phi \rangle| &= |\langle (1-\Delta)^{|s|} P\Phi, \partial_t \Phi \rangle| \\ &= |\langle P \underbrace{(1-\Delta)^{|s|} \Phi}_{\phi}, \partial_t \Phi \rangle| + |\langle [(1-\Delta)^{|s|}, P] \Phi, \partial_t \Phi \rangle| \end{aligned}$$

The right term has order $2|s| + 2 - 1$. Using duality,

$$\lesssim \|P\phi\|_{H^s} \|\partial_t \Phi\|_{H^{|s|}} \|\Phi\|_{H^{|s|+1}} \|\partial_t \Phi\|_{H^{|s|}}.$$

This completes the proof. □

1.3 Proof of well-posedness from the a priori estimate

Now we can quickly conclude the proof existence and uniqueness theorem.

Proof. Note that uniqueness and the a priori estimate follow from the proposition. It remains to prove existence.

Step 1: First, view this as trying to find the inverse of the operator $P : L^\infty_t([0,T], \mathcal{H}^{s+1}) \to L^1_t([0,T]; H^s)$. We want to reduce to the case when the initial data $g, h = 0$; we may achieve this using extension and modifying f.

Step 2: By duality, $\phi \in L^\infty_t([0,T]; H^{s+1}) = (L^1_t([0,T]; H^{-s-1}))^*$. We want

$$\begin{aligned} \int_0^T \langle f, \psi \rangle \, dt &= \int_0^T \langle P\phi, \psi \rangle \, dt \\ &= \int_0^T \langle \phi, P^*\psi \rangle \, dt. \end{aligned}$$

Define $\ell : P^*(L^1_t([0,T]; H^{-s})) \to \mathbb{R}$ by $\ell(P^*\psi) = \int_0^T \langle f, \psi \rangle \, dt$. This is well-defined by our a-priori estimate:

$$\|\ell\| \le \|f\|_{L^1(H^s)} \|\psi\|_{L^\infty(H^{-s})} \le \|f\|_{L^1(H^s)} \|P^*\psi\|_{L^1(H^{-s-1})}.$$

By Hahn-Banach, there exists an extension $\ell^* \in (L^1_t(H^{-s-1}))^*$ which is an extension with the bound $\|\ell^*\| \lesssim \|f\|_{L^1(H^s)}$. Here, $\phi = \ell^* \in L^\infty_t(H^{s+1})$.

Step 3: Upgrade $\phi \in L^\infty_t(H^{s+1})$ to $\phi \in C_t(H^{s+1})$ with $\partial_t \phi \in C_t(H^s)$. The way to do this is to approximate by smooth objects and try to take the limit. The a priori estimate will stay intact through the limit. □